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A Random Field Model for Anomalous Diffusion in Heterogeneous Porous Media

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Heterogeneity, as it occurs in porous media, is characterized in terms of a scaling exponent, or fractal dimension. A feature of primary interest for two-phase flow is the mixing length. This paper determines the relation between the scaling exponent for the heterogeneity and the scaling exponent which governs the mixing length. The analysis assumes a linear transport equation and uses random fields first in the characterization of the heterogeneity and second in the solution of the flow problem, in order to determine the mixing exponents. The scaling behavior changes from long-length-scale dominated to short-length-scale dominated at a critical value of the scaling exponent of the rock heterogeneity. The long-length-scale-dominated diffusion is anomalous.

KEY WORDS: Random fields; porous media; heterogeneity; anomalous diffusion.

1. INTRODUCTION

Random fields provide a natural description of rock heterogeneities, in the typical case in which the geological knowledge of the rock is much less detailed than is necessary to predict flow properties through it deterministically. Rock heterogeneities are a major mechanism governing the performance of enhanced oil recovery processes,(10) and they also play an important role in the ecology of pollutant transport in ground water.(13) These heterogeneities occur and produce important effects on all length scales. In the case of petroleum reservoirs, macroscopic heterogeneities result in the initiation of fingering instabilities which degrade the chemical,

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polymer, and miscible displacement processes of enhanced oil recovery. Even for the simplest oil–fluid displacement process (water displacing oil), they are primary factors limiting the total oil recovery. At the microscopic level, the statistical distributions of pore and throat geometries dominate the flow phenomena. A correct prediction of fluid flow through such porous media requires the integration of effects from many different length scales. This is called the scaleup problem.

The solution of the scaleup problem on the basis of numerical simulation at all length scales would require a detailed knowledge of rock heterogeneities, which is not available from feasible observations, as well as extensive computer resources.

We adopt an alternative approach to the scaleup problem in this paper. Rock heterogeneity, as it occurs in porous media, can be characterized approximately by a scaling exponent, or fractal dimension. Heterogeneities of this kind can be described in a natural way by multi-length-scale random fields. The analysis we give for the scaleup problem assumes a linear transport equation and uses random fields first in the characterization of the heterogeneity and second in the solution of the flow problem.

A feature of primary interest in two-phase flow is the mixing length. We show that the mixing length $l = l(t)$ between two fluids in heterogeneous media has an anomalous diffusion behavior $l = O(t^\alpha)$ for $1/2 \leqslant \alpha \leqslant 1$, where α is a scaling exponent which characterizes the mixing behavior, and which depends on the scaling properties of the rock heterogeneity. We also determine a critical value for the rock heterogeneity scaling behavior, which governs the transition from normal diffusion ($\alpha = 1/2$) to anomalous diffusion ($1/2 < \alpha$). The analysis is conducted in an approximation in which the heterogeneities serve to advance or retard the flow along a streamline, but the streamlines themselves are assumed to be straight lines, i.e., not affected significantly by the heterogeneities. Field observations are consistent with a travel-time- or distance-dependent diffusion coefficient $\nu \approx O(t^{0.75})$.[8,11,12] Since the mixing length $l(t)$ for the diffusion equation is $(\nu t)^{1/2}$, we have $l(t) \approx O(t^{1.7/2}) \approx O(t^{0.879})$, or $\alpha \approx 0.875$. The data show considerable scatter, indicating that the anomalous dimension, e.g., $\alpha \approx 0.875$, is not universal, but varies from reservoir to reservoir. In fact, α depends on the reservoir heterogeneity and especially on the degree to which the horizontal layering is (a) significant and (b) discontinuous or variable as a function of horizontal distance. Both of these geological properties vary from one reservoir to another, providing further justification of nonuniversality of α. In other contexts, anomalous diffusion results from a nonlinear diffusion equation[7] and from random velocity fields.[2]

2. A RANDOM FIELD DESCRIPTION OF POROUS MEDIA

Flow in porous media is governed by Darcy's law, which expresses the flow velocity as

$$v = -(k/\mu)\,\nabla P \tag{2.1}$$

Here P is the pressure, μ is the fluid viscosity, and $k = k(\mathbf{x})$ is the rock permeability. We model $k(\mathbf{x})$ with a log normal distribution, i.e., $\log k(\mathbf{x})$ is a Gaussian random field. Field data are usually fit well with a log normal distribution in the vertical direction. Horizontal variations of k are typically less extreme, and in the long-distance limit of concern to this paper, the correlation function is small. Thus, the distinction between normal and log normal distributions for horizontal variations in k does not show up in field data. For simplicity (i.e., to avoid the use of vertically scaled variables), we disregard the distinction in behavior between the horizontal and vertical directions, and treat the random field as isotropic and stationary.

Consider a linear transport law

$$u_t + v u_x = 0 \tag{2.2}$$

for transport of the relative volume fraction u of one of the fluids. An interface $x = x(y, z, t)$ between the fluids moves passively with the particle velocity v. Both u and x are random fields, depending on the Gaussian random field $\xi = \ln k - \langle \ln k \rangle$. Let $l(t) = \langle (x - \langle x \rangle)^2 \rangle^{1/2}$ be the mixing length. Central to our analysis of the scaling law for $l(t)$ is a decomposition of ξ into long- and short-length-scale components. This decomposition is time dependent, as is typical of situations to which the renormalization group applies. The crossover length scale separating short- from long-length-scale behavior is given by the expected travel distance $D(t) = \langle v \rangle t$. The long-distance fluctuations add coherently in each realization ξ of the random field. Short-distance fluctuations add incoherently. To understand them, we replace space averages by ensemble averages, and estimate $l(t)$ in terms of a random walk. Let

$$\langle \xi(\mathbf{x})\, \xi(\mathbf{y}) \rangle = O(|\mathbf{x} - \mathbf{y}|^{-\beta}) \tag{2.3}$$

for $\beta \geqslant 0$. Our main result is

$$l(t) = O(t^{\alpha}) \tag{2.4}$$

where $\alpha = \max\{1/2, 1 - \beta/2\}$. It follows that the critical value β_{cr} of the rock heterogeneity exponent β governing the transition between normal and anomalous diffusion is $\beta_{\text{cr}} = 1$.

To simplify the analysis, we represent ξ as a hierarchical random field, in d space dimensions.[5] Associated with ξ is a sequence of lattices $2^n Z^d$, indexed by n and with spacing 2^n. We introduce the Hilbert space $H_n = L_2(2^n Z^d)$. With $f, g \in H_n$ interpreted as functions on R^d which are piecewise constant on mesh blocks, H_n has the inner product $\langle f, g \rangle = \sum f_i \bar{g}_i$. Let i and j be vectors in $2^n Z^d$. A scaled white noise covariance C_n on H_n is

$$C_n = 2^{-n\beta} I_{H_n} = 2^{-n\beta} \delta_{ij} \tag{2.5}$$

By the general theory,[6] C_n defines a Gaussian measure and a cutoff Gaussian field ξ_n which takes on a constant value on each of the lattice squares of the lattice $2^n Z^d$. We define $\xi = \sum \xi_n$. The correlation function of ξ is

$$\begin{aligned} \langle \xi(r_1)\, \xi(r_2) \rangle &= \sum_{2^n \geq |r_1 - r_2|} \langle \xi_n(r_1)\, \xi_n(r_2) \rangle \\ &\approx \sum_{2^n \geq |r_1 - r_2|} 2^{-n\beta} = O(|r_1 - r_2|^{-\beta}) \end{aligned} \tag{2.6}$$

for "typical" points r_1 and r_2. This is the desired asymptotic behavior of the correlation function.

It is also of interest to present an alternative description of a random field ψ which is fully scale invariant.[8] We first discuss specification of boundary conditions at well locations and at the reservoir boundary. We define a generalized fractal free field in terms of the Fourier transform $\hat{f}_2$ of its two-point function f_2:

$$\begin{aligned} \hat{f}_2(k) &= \int dx\, e^{ikx} \langle \psi(x)\, \psi(0) \rangle \\ &= \int (k^2 + m^2)^{-\gamma}\, d\rho(m) \end{aligned} \tag{2.7}$$

In this definition, generalized refers to the arbitrary combination of length scales contained in the measure $d\rho$, and fractal refers to the arbitrary power law behavior given by the exponent γ. For $\gamma = 1$, $m^2 = 0$, i.e., $d\rho(m) = \delta(m)\, dm$ and in one space dimension, ψ defines a Wiener process, having independent increments. That is, increments in observations made at different locations are uncorrelated. Field observations suggest, on the contrary, the existence of statistically significant trends. Following ref. 9, we represent these trends by choosing $\gamma > 1$ and $m^2 = 0$, for one-dimensional correlations along streamlines, and we now discuss the generalization of this idea to higher space dimensions.

Imposition of boundary conditions in the form of point values at well locations on a discontinuous process is meaningless. The continuity properties of ψ are therefore important. Wiener processes ($\gamma = 1$) in space dimensions $d > 1$ are discontinuous. In fact, the set of continuous sample paths is of measure zero in the set of all sample paths. Moreover, the sample paths which are functions have measure zero, and the typical sample path is a Schwartz distribution of negative order. However, for $\gamma > 1$ in two space dimensions, the random field ψ is continuous, for a.e. realization of ψ.[3,4] In order to have the one-dimensional sections (as would be observed along an outcrop) agree with the above one-dimensional fractal Brownian motion, we choose a covariance $|k|^{-2\gamma-(d-1)}$; this will have the same regularity properties on one-dimensional sections (such as outcrops or streamlines) as the one-dimensional fractal Brownian motion with covariance $|k|^{-2\gamma}$. This process has continuous sample paths in all dimensions.

Thus, boundary conditions are imposed by constraining the values of ψ to agree with observed values at well locations, and to vanish at the reservoir boundary. This constrained process ψ is related to the fractal free field or Wiener process by a change of variables in function space, involving the solution of a pseudodifferential equation. Let φ denote the white noise process, with covariance operator I in Fourier space. The mapping T which connects ψ to ϕ via the formula $\psi = T\phi$ transforms the covariance C (which is the identity operator I for ϕ) to TCT^*, the covariance of ψ. For translation-invariant operators, given by convolution kernels, the operator product is a convolution product. Thus, to achieve the specified covariance for ψ, we choose as the mapping function T from ϕ to ψ the convolution square root of the ψ covariance, which is thus $|k|^{-\gamma-(d-1)/2}$ in Fourier space and $|x|^{-\gamma-(d+1)/2}$ in position space. In the case of boundary conditions, these full-space Green's functions must be modified so that the covariance satisfies boundary conditions, and the mapping between random fields is the operator square root of the covariance, which now does not have a simple expression in either position space or Fourier space. We note for later use that the mapping from white noise ϕ to the reservoir variable (log permeability) ψ is either growing with x ($d = 1$) or decreasing slowly ($d = 2, 3$), so that in all cases the integral along streamlines diverges. The growth for large x of log permeability in this model is meaningless, and is resolved by use of boundary conditions, so that values of ψ are given at well locations and then interpolated with the constrained stochastic process between.

3. THE MIXING LENGTH

Consider the linear transport law (2.2), with velocity defined by Darcy's law (2.1) and incompressibility, $\nabla \cdot \mathbf{v} =$ source terms. This system is known as miscible displacement. We take $\mu_1 = \mu_2$, where μ_i is the viscosity for the ith phase, and then the system is neutrally stably with respect to fingering instabilities. As boundary conditions, consider a flow predominantly parallel to the x axis, with a pressure difference $[P] = P|_{x=0} - P|_{x=X}$ specified between the domain faces $x=0$ and $x=X$, and with no flow boundary conditions across the other four faces $y=0$, $y=Y$, $z=0$, and $z=Z$ of a rectangular domain.

We analyze separately the long- and short-length-scale fluctuations in ξ. Consider first the long-length-scale fluctuations. Asymptotically at large distances, $\langle \xi_n^2 \rangle = O(2^{-n\beta})$ is small and the log normal distribution can be approximated for our purposes by a normal distribution. In particular, $\langle k \rangle = e^{\langle \ln k \rangle}$ as far as the large length scale fluctuations are concerned. In this limit, we set $\nabla \xi = 0$ and regard ∇P as a nonrandom variable, which takes on a contant value. Then

$$v = -\frac{(\langle k \rangle e^{\xi})}{\mu} \nabla P = -\frac{\langle k \rangle (1+\xi)}{\mu} \nabla P = v_0 + \delta v \tag{3.1}$$

At time t, the expected travel distance is $\bar{x} = v_0 t$, and so the long range part of ξ is $\sum_{\{n: 2^n > v_0 t\}} \xi_n$. Let $\delta x = x - \bar{x}$. Then

$$2l(t)\frac{dl(t)}{dt} = \frac{d}{dt} l(t)^2 = 2\langle \delta x \, \delta v \rangle$$

In the long-distance limit, the random fluctuations in ξ_n contribute coherently to δx and δv. Thus, δx is proportional to δv, and

$$l\frac{dl}{dt} = t\langle \delta v^2 \rangle = t v_0^2 \langle \xi^2 \rangle$$
$$= 2^{\beta} v_0^2 t 2^{-n\beta} = 2^{\beta} v_0^2 (v_0 t)^{-\beta} t = 2^{\beta} v^{2-\beta} t^{1-\beta}$$

Writing $l = O(t^{\alpha})$, we substitute, and determine that $\alpha = 1 - \beta/2$, i.e., $l = O(t^{1-\beta/2})$.

Next we consider the effect of the short-length-scale portion $\sum_{\{n: 2^n < \bar{x}\}} \xi_n$ of the heterogeneity field. The length scales in question are large relative to unity, but small relative to the travel distance. In the approximation $2^n \ll \bar{x}$, short-scale fluctuations add incoherently, and we can replace space averages by ensemble averages. In this regime, ξ influences the front x as in a random walk, and $O(t^{1/2})$ behavior results.

However, the time t in $O(t^{1/2})$ is scaled by the length of the perturbation and its covariance, as we now see. Moreover, in the short-length-scale regime, ∇P cannot be regarded as a classical, or nonrandom, variable.

The pressure P is defined as the solution of the elliptic equation

$$LP = \nabla \cdot \left(\frac{-k}{\mu} \nabla P \right) = \text{source}$$

Let

$$G = L^{-1}, \qquad L_0 = \nabla \cdot \left(\frac{-\bar{k}}{\mu} \nabla \right), \qquad \delta L = \nabla \cdot \left(\frac{-\bar{k}\xi}{\mu} \nabla \right), \qquad G_0 = L_0^{-1}$$

We assume ξ is small, since we are interested in the large-distance limit. To first order in ξ, $\delta G = -G_0\, \delta L G_0$. Let $S_0 = \nabla G_0 \nabla$, which in Fourier space has the expression $(k \otimes k)/k^2$. Then

$$\delta v = -S_0 \xi v_0$$

Note that S_0 is a singular integral operator of order zero, defined by a Cauchy principal value, and decays like r^{-d} as $r \to \infty$. We see that the influence of ξ in a single heterogeneity mesh block can be integrated along streamlines and is localized near that block in both the streamwise and transverse directions.

In computing $\langle \delta x^2 \rangle$ we note that δx is a sum of terms depending on the ξ coming from different heterogeneity mesh blocks and lattices. In this sum, all cross terms drop out, because the summands contributing to ξ are independent. Thus, we can hold fixed the exponent n which characterizes the length scale and introduce the mean transit time $t_n = 2^n/v_0$ across a mesh block in the 2^n lattice. The contribution to δx from ξ in a single mesh block is

$$\int -S_0 \xi v_0 \, dt$$

where the integral is taken in time, along a streamline.

For a given ξ_n mesh block, the covariance of the fluctuating portion δx is

$$\langle \delta x^2 \rangle = O(2^{n(2-\beta)})$$

According to the central limit theorem, $l(t)$ is asymptotic to $\sigma \sqrt{N}$, where $N = t v_0 2^{-n}$ is the number of independent mesh blocks contributing to this sum and $\sigma = \langle \delta x \rangle^{1/2}$. This argument does not require the field ξ to be Gaussian. Thus,

$$l(t) \approx [(tv_0 2^{-n})(2^{n(2-\beta)})]^{1/2} = v_0^{1/2} t^{1/2} 2^{n(1/2-\beta/2)}$$

For $0 \leqslant \beta \leqslant 1$, we can evaluate at the maximum short-range value of n, so that $2^n = v_0 t$ and

$$l(t) = O(v_0^{1-\beta/2} t^{1-\beta/2})$$

while for $\beta \geqslant 1$, the maximum short-range value effect occurs for $n = O(1)$. In this case $l(t) = O(t^{1/2})$, which is the conventional diffusion limit.

4. DISCUSSION AND CONCLUSION

To model anomalous diffusion processes at the continuum level, as predicted in the previous section, we consider the equation

$$u_t + vu_x = D(u) \tag{4.1}$$

where D is a linear differential or pseudodifferential operator. We discuss two approaches to determine D. The conventional approach is to take $D = \nu\Delta$, where the diffusion "constant" ν is a function of the flow history, and in particular of the travel time or distance along stream lines. For example, the relation $\nu = O(t^{0.75})$ was discussed above, on the basis of field data. To avoid history dependence, and to obtain a properly posed initial value problem for time integration of a differential equation, it is necessary to enlarge the set of state variables. For example, flow distance along streamlines could be regarded as a new dependent variable, and the system of equations enlarged to include this variable.

The derivation of Section 3 supports the notion of history-dependent diffusion. The diffusion results from an averaging process. The formalism of compensated compactness[1] would appear to be useful in this connection. If the averaging is over an ensemble of reservoirs, then there is no history dependence. All length scales then contribute to the averaging process for all time. Normally this is not what is wanted. If the averaging is over space, for a single reservoir selected from an ensemble, then only small-scale fluctuations (on a scale less than the travel distance, which is normally larger than the averaging length scale) contribute to the average, while large-scale fluctuations contribute to a coherent uncertainty in frontal position associated with the selection of a single reservoir from the ensemble. To see that Section 3 leads to this picture, we note that the generator of a sum of independent diffusion processes is the sum of the generators. Similarly, the convolution product of the Gaussian solution operators is still a Gaussian. The above reasoning determines which terms or factors to include in the sum or product. For spatial averages, the result is history dependent, as discussed above.

Spatially averaged anomalous scaling behavior can be obtained when D is a pseudodifferential operator of fractional order. Pseudodifferential operators in time of order 1/2 have been used to model transients in diffusion for pipeline flow.[14] Scaling arguments show that in order to obtain anomalous diffusion, $l(t) \sim O(t^{\alpha})$, $1/2 < \alpha < 1$, we should choose D to have order $1/\alpha$. A translation, $x \to x - vt$, into comoving coordinates eliminates the vu_x term from (4.1) and yields $u_t = D(u)$. Anomalous scaling, $x \to ax$, $t \to a^{1/\alpha}t$, now shows that the order of D should be chosen as $1/\alpha$.

It is possible to use local, but nonlinear, differential operators (as opposed to pseudodifferential operators) to obtain anomalous diffusion in the continuum equations.[7] The analysis of ref. 2 is closer methodologically to ours (the two efforts were conducted independently), but differs in detail; in particular, the heterogeneity models proposed there lead to a single value for the anomalous diffusion exponent.

If averages over a fixed (small) length scale (or conceptually point values of the solution) are desired, as is required for the correct treatment of most nonlinear flow processes, then only the small-length-scale diffusion is used, $l = O(t^{1/2})$, and the multi-length-scale heterogeneous aspects of the flow mixing behavior must be described by additional state variables, leading to an enlarged system, such as the dual porosity models used to describe fractured reservoirs.

To summarize, we have shown that multi-length-scale random fields are useful for modeling petroleum reservoir heterogeneity and the associated fluid flow. We provide a theoretical explanation for the observed anomalous diffusion. We found the critical scaling exponent $\beta_{\mathrm{cr}} = 1$ of the rock heterogeneity for the crossover from regular to anomalous diffusion. Further study is needed not only to amplify the solution proposed here, but to examine the hypotheses proposed. There is, for example, no reason for expecting geological variability to be characterized by a one-parameter family of Gaussian measures, specified by a single scaling law.

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